A STOCHASTIC PARTICLE METHOD FOR THE SOLUTION OF A 1D VISCOUS SCALAR CONSERVATION LAW IN A BOUNDED INTERVAL

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Abstract

We give a probabilistic interpretation of a viscous scalar conservation law in a bounded interval thanks to a nonlinear martingale problem. The underlying nonlinear stochastic process is reflected at the boundary to take into account the Dirichlet conditions. After proving uniqueness for the martingale problem, we show existence thanks to a propagation of chaos result. Indeed we exhibit a system of N interacting particles, the empirical measure of which converges to the unique solution of the martingale problem as $N \to +\infty$. As a consequence, the solution of the viscous conservation law can be approximated thanks to a numerical algorithm based on the simulation of the particle system. When this system is discretized in time thanks to the Euler-Lépingle scheme [10], we show that the rate of convergence of the error is in $\mathcal{O}(\Delta t + 1/\sqrt{N})$ where Δt denotes the time step. Finally, we give numerical results which confirm this theoretical rate.

Introduction

We are interested in the following viscous scalar conservation law with non homogeneous Dirichlet boundary conditions on the interval $[0,1]$:

$$\begin{cases} \frac{\partial}{\partial t}v(t,x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}v(t,x) - \frac{\partial}{\partial x}A(v(t,x)), \forall (t,x) \in (0,+\infty)\times(0,1) \\ \forall x \in [0,1],\ v(0,x) = v_0(x), \\ \forall t \geq 0,\ v(t,0) = 0 \text{ and } v(t,1) = 1, \end{cases} \tag{1}$$

We suppose that $A : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 function and that the initial data v_0 is the cumulative distribution function of a probability measure U_0 on $[0,1]$, which writes $v_0(x) = \int_0^x U_0(dy) = H * U_0(x)$ where $H(y) = 1_{\{y\geq 0\}}$ denotes the Heaviside function.

When the viscous scalar conservation law is posed in the spatial domain $\mathbb{R}$ instead of $[0,1]$, one can show that its unique weak solution is equal to $H * P_t(x)$ where $(P_t)_{t\geq 0}$ denote the time-marginals of a probability measure P on the space of continuous sample paths $C([0,+\infty),\mathbb{R})$ characterized by a martingale problem nonlinear in the sense of McKean [4] [8].

Here, we follow a similar approach. To take into account the Dirichlet boundary conditions, we work with a diffusion process with reflection. That is why we introduce (X,K) the canonical process on the sample path space $\mathcal{C} = C([0,+\infty),[0,1]) \times C([0,+\infty),\mathbb{R})$ (endowed with the topology of uniform convergence on compact sets). For P in the set $\mathcal{P}(\mathcal{C})$ of probability measures on $\mathcal{C}$, let $(\bar{P}_t)_{t\geq 0}$ denote the time-marginals of the probability measure $\bar{P}$ on $C([0,+\infty),[0,1])$ defined by $\bar{P} = P \circ X^{-1}$. We associate the following nonlinear problem with (1)

Definition 0.1 *A probability measure $P \in \mathcal{P}(\mathcal{C})$ solves the martingale problem (MP) starting at $U_0 \otimes \delta_0 \in \mathcal{P}([0,1] \times \mathbb{R})$, if*

i) $P \circ (X_0, K_0)^{-1} = U_0 \otimes \delta_0$

ii) $\forall \varphi \in C_b^2(\mathbb{R}),\ \varphi(X_t - K_t) - \varphi(X_0 - K_0) - \int_0^t \frac{\sigma^2}{2}\varphi''(X_s - K_s) + A'(H * \bar{P}_s(X_s))\varphi'(X_s - K_s)ds$ *is a P martingale*

iii) P *a.s.,* $\forall t \geq 0,\ \int_0^t d|K|_s < +\infty,\ \ |K|_t = \int_0^t 1_{\{0,1\}}(X_s)d|K|_s$ *and* $K_t = \int_0^t (1-2X_s)d|K|_s$.

In the first section, we prove that if P solves problem (MP), then $(t,x) \to H * \bar{P}_t(x)$ is the unique weak solution of (1). We deduce uniqueness for the martingale problem. Existence is obtained thanks to a propagation of chaos result for a system of weakly interacting diffusion processes. In section 2, we discretize this system in time thanks to the version of the Euler scheme introduced by Lépingle [10]. This way, we derive a numerical method to approximate the solution of (1). We prove a theoretical rate of convergence in $\mathcal{O}(\Delta t + 1/\sqrt{N})$ which is confirmed by numerical experiments. This rate is the same as the one obtained by Bossy [2] when the spatial domain is $\mathbb{R}$. The treatment of the reflection by the Euler Lépingle scheme does not alter the convergence whereas we exhibit a sublinear dependence on the time step Δt in numerical experiments relying on the cruder Euler projection scheme.

1 Probabilistic interpretation of the viscous scalar conservation law

1.1 Uniqueness for the martingale problem (MP) and link with equation (1)

For $\mathcal{Q}_T = (0,T) \times (0,1)$, let $W_2^{0,1}(\mathcal{Q}_T)$ and $W_2^{1,1}(\mathcal{Q}_T)$ denote the Hilbert spaces with respective scalar products (cf. [9])

$$\begin{aligned}(u,v)_{W_2^{0,1}(\mathcal{Q}_T)} &= \int_{\mathcal{Q}_T}(uv + \partial_x u \partial_x v)dxdt,\\ (u,v)_{W_2^{1,1}(\mathcal{Q}_T)} &= \int_{\mathcal{Q}_T}(uv + \partial_x u \partial_x v + \partial_t u \partial_t v)dxdt.\end{aligned}$$

We introduce the Banach space $V_2^{0,1}(\mathcal{Q}_T) = \{u \in W_2^{0,1}(\mathcal{Q}_T) \cap C((0,T), L^2(0,1))$ such that $\|u\|_{V_2^{0,1}(\mathcal{Q}_T)} = \sup_{0\leq t\leq T}\|u(t,x)\|_{L^2(0,1)} + \|\partial_x u\|_{L^2(\mathcal{Q}_T)} < +\infty\}$. The subspaces of these three spaces consisting in elements which vanish on $[0,T]\times\{0,1\}$ are respectively denoted by $\overset{\circ}{W}{}_2^{0,1}(\mathcal{Q}_T)$, $\overset{\circ}{W}{}_2^{1,1}(\mathcal{Q}_T)$, $\overset{\circ}{V}{}_2^{0,1}(\mathcal{Q}_T)$.

We first prove uniqueness of weak solutions of problem (1) defined in the following way :

Definition 1.1 *A weak solution of (1) is a function $v : [0,+\infty) \times [0,1] \to \mathbb{R}$ satisfying the boundary conditions and such that for any $T > 0$, $v \in V_2^{0,1}(\mathcal{Q}_T) \cap L^\infty(\mathcal{Q}_T)$ and for all ϕ in $\overset{\circ}{W}{}_2^{1,1}(\mathcal{Q}_T)$ and all t in $[0,T]$,*

$$\begin{aligned}\int_0^1 v(t,x)\phi(t,x)dx &= \int_0^1 v_0(x)\phi(0,x)dx + \int_0^t \int_0^1 \frac{\partial}{\partial s}\phi(s,x)v(s,x)dxds \\ &+ \int_0^t \int_0^1 \frac{\partial}{\partial x}\phi(s,x)A(v(s,x))dxds \\ &- \int_0^t \int_0^1 \frac{\sigma^2}{2}\frac{\partial}{\partial x}\phi(s,x)\frac{\partial}{\partial x}v(s,x)dxds.\end{aligned} \tag{2}$$

Lemma 1.2 *Equation (1) has no more than one weak solution in the sense of Definition 1.1.*

The proof of Lemma 1.2 can be found in [3]. The difference $w = v^1 - v^2$ of two weak solutions is a generalized solution in the sense of Ladyzenskaja, Solonnikov and Ural'ceva (cf. [9]) of a linear equation with uniformly bounded coefficients. We follow techniques from [9] chapter 3 to show that $\|w\|_{V_2^{0,1}(\mathcal{Q}_T)} = 0$.

Next, we check that if P solves the martingale problem (MP), then $V(t,x) = H * \bar{P}_t(x)$ is a weak solution of (1).

Proposition 1.3 *If P solves the martingale problem (MP) starting at $U_0 \otimes \delta_0$, then $V(t,x) = H * \bar{P}_t(x)$ is a weak solution of (1). Moreover, uniqueness holds for the martingale problem (MP).*

Sketch of the proof : the function $V(t,x) = H * \bar{P}_t(x)$ is bounded by 1 and satisfies $V(t,1) = 1$. We check that $V(t,0) = 0$ and that the function V belongs to $V_2^{0,1}(\mathcal{Q}_T)$ for any $T > 0$ thanks to the following Lemma which is deduced from estimations of the density of the doubly reflected Brownian motion on $[0,1]$ by using Girsanov theorem.

Lemma 1.4 [3]. *If P solves the martingale problem (MP) then for any $t > 0$, $\bar{P}_t$ has a density $\bar{p}_t$ which belongs to $L^2([0,1])$ and $\|\bar{p}_t\|_{L^2([0,1])} \leq C\left(1 + t^{-1/4}\right)\exp(Ct)$.*

We still have to check that V satisfies the identity (2). By density it is enough to suppose that the test function ϕ is C^∞ on $[0,T] \times [0,1]$ and satisfies $\phi(t,0) = \phi(t,1) = 0$ for all $t \in [0,T]$. We set $\psi(t,x) = \int_0^x \phi(t,y)dy$. Then ψ is C^∞ with $\frac{\partial \psi}{\partial x}(t,0) = \frac{\partial \psi}{\partial x}(t,1) = 0$. By Definition 0.1 ii, under the probability measure P, $\frac{1}{\sigma}(X_t - X_0 - K_t - \int_0^t A'(H * \bar{P}_s(X_s))ds)$ is a local martingale with quadratic variation t i.e. a Brownian motion. To deduce (2), we compute $\psi(t, X_t)$ by Itô's formula, take expectations in the obtained relation, use the equality $\bar{p}_s = \frac{\partial V}{\partial x}(s,.)$ and apply Stieljes integration by parts formula in some of the spatial integrals.

Uniqueness for the martingale problem (MP) is derived from the uniqueness result for problem (1): if P and Q solve (MP), then for any $(t,x) \in [0,+\infty) \times \mathbb{R}, H * \bar{P}_t(x) = H * \bar{Q}_t(x)$. Hence P and Q solve a linear martingale problem with bounded drift term $A'(H * \bar{P}_t(x))$ and by Girsanov theorem, $P = Q$.

The probabilistic interpretation is completed by a propagation of chaos result which ensures existence for problem (MP).

1.2 The propagation of chaos result

The system of weakly interacting particles with normal reflecting boundary conditions is given by the stochastic differential equation :

$$\begin{cases} X_t^{i,N} = X_0^{i,N} + \sigma W_t^i + \int_0^t A'(H * \bar{\mu}_s^N(X_s^{i,N}))ds + K_t^{i,N} \\ |K^{i,N}|_t = \int_0^t 1_{\{0,1\}}(X_s^{i,N})d|K^{i,N}|_s, \quad K_t^{i,N} = \int_0^t (1-2X_s^{i,N})d|K^{i,N}|_s, \quad i \leq N \end{cases} \tag{3}$$

where $\bar{\mu}_s^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,N}}$ and $(W^1, \ldots, W^N)$ is a N-dimensional Brownian motion independent of the initial variables $(X_0^{1,N}, \ldots, X_0^{N,N})$ which are I.I.D. with law U_0.
As $\sup_{[0,1]}|A'(x)|$ is bounded, by Girsanov theorem, this equation admits a unique weak solution.

Theorem 1.5 *The particle systems $((X^{1,N}, K^{1,N}), \ldots, (X^{N,N}, K^{N,N}))$ are P-chaotic where P denotes the unique solution of the martingale problem (MP) starting at $U_0 \otimes \delta_0$ i.e. for fixed $j \in \mathbb{N}^*$ the law of $((X^{1,N}, K^{1,N}), \ldots, (X^{j,N}, K^{j,N}))$ converges weakly to $P^{\otimes j}$ as $N \to +\infty$ or equivalently the empirical measures μ^N considered as $\mathcal{P}(\mathcal{C})$-valued r.v. converge in probability to the constant P.*

As an easy consequence, it is possible to approximate the weak solution $V(t,x) = H * \bar{P}_t(x)$ of (1) thanks to the empirical cumulative distribution function $H * \bar{\mu}_t^N(x)$ of the particle system. More precisely,

Corollary 1.6 $\forall (t,x) \in [0,+\infty) \times [0,1]$, $\lim_{N\to+\infty} \mathbb{E}|V(t,x) - H * \bar{\mu}_t^N(x)| = 0$.

The proof of Theorem 1.5 given in [3] follows the one given by Sznitman [11] Theorem 1.4 except in the treatment of the discontinuity of the Heaviside function. When possible, we take advantage of the particular form of our spatial domain (the interval $[0,1]$) to simplify the arguments.

2 Particle method

2.1 The algorithm and its theoretical convergence rate

According to Corollary 1.6, it is possible to approximate $V(t,x)$ by the empirical cumulative distribution function $H * \bar{\mu}_t^N(x) = \frac{1}{N}\sum_{j=1}^N H(x - X_t^{j,N})$ of the particle system (3). The particle method consists in simulating the particle system (3).

We discretize in time the N-dimensional stochastic differential equation (3) thanks to the version of the Euler scheme introduced by Lépingle [10] which mimics the reflection at the boundary. We choose $\Delta t > 0$ and $L \in \mathbb{N}$ such that $T = L\Delta t$ and denote by $Y_{t_l}^i$ the position of the i-th particle $(1 \leq i \leq N)$ at the discretization time $t_l = l\Delta t$ $(0 \leq l \leq L)$. The Euler-Lépingle scheme consists in setting $0 < \alpha_0 < \alpha_1 < 1$ and in generating exact reflection on the lower-boundary on $[t_l, t_{l+1}]$ when $Y_{t_l}^i \leq \alpha_0$ and exact reflection on the upper-boundary on $[t_l, t_{l+1}]$ when $Y_{t_l}^i \geq \alpha_1$. The other cases of reflection are treated by projection onto $[0,1]$. We invert the initial cumulative distribution function $V_0(x) = H * U_0(x)$ to construct the set of initial positions of the numerical particles :

$$y_0^i = \inf\left\{z : H * U_0(z) \geq \frac{i}{N}\right\} \quad \text{for} \quad 1 \leq i \leq N. \tag{4}$$

At time t_l, the function $V(t_l,x)$ is approximated thanks to the empirical cumulative distribution function

$$\overline{V}(t_l,x) = \frac{1}{N}\sum_{j=1}^{N} H(x - Y_{t_l}^j)$$

and the successive positions of the ith particle are given inductively by

$$\begin{cases} Y_{t_0}^i = y_0^i \\ Y_{t_{l+1}}^i = 0 \vee \left(Y_{t_l}^i + \sigma(W_{t_{l+1}}^i - W_{t_l}^i) + A'(\overline{V}(t_l, Y_{t_l}^i))\Delta t + C_{l+1}^i\right) \wedge 1 \\ C_{l+1}^i = \mathbb{1}_{\{Y_{t_l}^i \le \alpha_0\}} \sup\limits_{s\in[t_l,t_{l+1}]} \left(Y_{t_l}^i + \sigma(W_s^i - W_{t_l}^i) + (s-t_l)A'(\overline{V}(t_l, Y_{t_l}^i))\right)^- \\ \qquad - \mathbb{1}_{\{Y_{t_l}^i \ge \alpha_1\}} \sup\limits_{s\in[t_l,t_{l+1}]} \left(Y_{t_l}^i - 1 + \sigma(W_s^i - W_{t_l}^i) + (s-t_l)A'(\overline{V}(t_l, Y_{t_l}^i))\right)^+ . \end{cases} \tag{5}$$

Since it is possible to simulate jointly the Brownian increment $(W_{t_{l+1}}^i - W_{t_l}^i)$ and the corresponding $\sup_{s\in[t_l,t_{l+1}]}\left(W_s^i - W_{t_l}^i + (s-t_l)a\right)$ where a is a real constant (see [10]), this discretization scheme is feasible.

To obtain the optimal rate of convergence $\mathcal{O}(1/\sqrt{N} + \Delta t)$ we strengthen our hypotheses :

$$(H) \begin{cases} v_0 \in C^{2+\beta}([0,1]) \ (C^2 \text{ with } v_0'' \text{ Hölder continuous with exponent } \beta) \text{ where } \beta \in]0,1[, \\ \sigma^2 v_0''(0) = 2A'(0)v_0'(0) \quad \text{and} \quad \sigma^2 v_0''(1) = 2A'(1)v_0'(1), \\ A \text{ is a } C^3 \text{ function.} \end{cases}$$

These hypotheses are possibly too restrictive but they avoid further complications of the already technical proof. In particular, under (H), the weak solution of (1) is actually a classical solution :

Lemma 2.1 *Under hypothesis (H), the solution $V(t,x) = H * \bar{P}_t(x)$ of (1) belongs to $C^{1,2}([0,T]\times[0,1])$ and $\partial_x V(t,x)$ is Hölder continuous with exponent $(1+\beta)/2$ in the time variable t on $[0,T]\times[0,1]$.*

Our estimate of the convergence rate of the particle method is the following one :

Theorem 2.2 *Under hypothesis (H), there exists a strictly positive constant C depending on $A, U_0, T, \sigma, \alpha_0$ and α_1 such that*

$$\forall 0 \le l \le L, \ \sup_{x\in[0,1]} \mathbb{E}|V(t_l,x) - \overline{V}(t_l,x)| \le C\left(\frac{1}{\sqrt{N}} + \Delta t\right).$$

The proof given in [3] follows the main ideas of Bossy [2] who deals with the convergence rate of a particle approximation for the solution of the scalar conservation law with spatial domain $\mathbb{R}$ similar to (1) even if some new difficulties arise in the present framework because of the reflection. The nonlinearity of the problem prevents us to isolate the time discretization error from the global error produced by the algorithm. Nevertheless, the analysis of the weak error corresponding to the discretization of the stochastic differential equation with normal reflection

$$\begin{cases} X_t^y = y + \sigma W_t + \int_0^t b(s, X_s^y)ds + K_t^y \\ |K^y|_t = \int_0^t \mathbb{1}_{\{0,1\}}(X_s^y) d|K^y|_s, \quad K_t^y = \int_0^t (1 - 2X_s^y) d|K^y|_s \end{cases}$$

by the Euler-Lépingle scheme

$$\begin{cases} \widetilde{X}^y_{t_0} = y \\ \widetilde{X}^y_{t_{l+1}} = 0 \vee \left(\widetilde{X}^y_{t_l} + \sigma(W_{t_{l+1}} - W_{t_l}) + b(t_l, \widetilde{X}^y_{t_l})\Delta t + C_{l+1}\right) \wedge 1 \\ C_{l+1} = \mathbb{1}_{\{\widetilde{X}^y_{t_l} \leq \alpha_0\}} \sup_{s\in[t_l,t_{l+1}]} \left(\widetilde{X}^y_{t_l} + \sigma(W_s - W_{t_l}) + b(t_l, \widetilde{X}^y_{t_l})(s - t_l)\right)^- \\ \qquad -\mathbb{1}_{\{\widetilde{X}^y_{t_l} \geq \alpha_1\}} \sup_{s\in[t_l,t_{l+1}]} \left(\widetilde{X}^y_{t_l} - 1 + \sigma(W_s - W_{t_l}) + b(t_l, \widetilde{X}^y_{t_l})(s - t_l)\right)^+ \end{cases}$$

is an important step in the proof. Although limited to the case of a constant diffusion coefficient, this study is also interesting by itself since to our knowledge, the weak behaviour of the Euler-Lépingle has not been investigated yet. Assuming a regularity condition on the drift coefficient $b(s,x)$ which is satisfied by $A'(V(s,x))$ under hypothesis (H) (see Lemma 2.1), we upper-bound the weak convergence rate :

Proposition 2.3 *Assume that b is $C^{1,2}$ on $[0,T] \times [0,1]$ and that for some $\gamma > 0$, $\partial_x b(t,x)$ is Hölder continuous with exponent γ in t. Then there is a constant C depending on $\sigma, T, b, \alpha_0, \alpha_1$ but not on x and Δt such that when $f : [0,1] \to \mathbb{R}$ is a function with bounded variation and m denotes its distribution derivative,*

$$\forall l \leq L, \ \left|\mathbb{E}\left(f(X^y_{t_l}) - f(\widetilde{X}^y_{t_l})\right)\right| \leq C\Delta t \int_0^1 |m|(dx).$$

2.2 Numerical experiments

As a numerical benchmark, we consider the following Dirichlet problem for the viscous Burgers equation which corresponds to the choice $A(x) = x^2/2$:

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) - v(t,x)\dfrac{\partial v}{\partial x}(t,x), t > 0, x \in [0,2\pi] \\ v(0,x) = \dfrac{2\sin(x)}{\cos(x) + \mathrm{e}}, x \in [0,2\pi] \text{ and } \forall t \geq 0, \ v(t,0) = 0, v(t,2\pi) = 0, \end{cases} \tag{6}$$

The exact solution is (see [1]) $v(t,x) = 2\sin(x)/(\cos(x) + \mathrm{e}^{(1+t)})$.
The spatial domain $[0,2\pi]$ is different from the one considered so far but our results remain true for any bounded interval replacing $[0,1]$. The fact that the distribution derivative $m_0(x)dx$ of the initial data $v(0,x)$ given by $m_0(x) = (2 + 2\mathrm{e}\cos(x))/(\cos(x) + \mathrm{e})^2$ is not a probability measure but a bounded signed measure represents a more significant modification. In fact, we could not find any explicit solution when $v(0,.)$ is the cumulative distribution function of a probability measure.
To take into account this modification, we use weighted particles $(Y^i_{t_l}, w^i)_{1\leq i\leq N}$ (see for instance [8] which deals with a spatial domain equal to $\mathbb{R}$). The N initial locations $y^i_0 = \inf\{y; H * |m_0|/\|m_0\|_{L^1([0,2\pi])}(y) = \frac{i}{N}\}$ are chosen in order to approximate the cumulative distribution function of the probability measure $|m_0|(x)dx/\|m_0\|_{L^1([0,2\pi])}$ and the corresponding weights are $w_i = \|m_0\|_{L^1([0,2\pi])}\mathrm{sign}(m_0(y^i_0))$. The approximate solution is given by the weighted cumulative distribution function of the particle system $\overline{V}(t_l,x) = \frac{1}{N}\sum_{i=1}^N w^i H(x - Y^i_{t_l})$ where the successive positions are defined inductively by (5) but with $\wedge 1$ (resp. -1) replaced by $\wedge 2\pi$ (resp. -2π) in the second (resp. last) line.
The parameters of Lépingle scheme are $\alpha_0 = 0.25$ and $\alpha_1 = 2\pi - 0.25$. We have plotted on Figure 1 the numerical solution at time $t = 1$. As the dependence of the error on the number of particles is standard and corresponds to the usual central limit theorem rate (see [4][5][8] for numerical

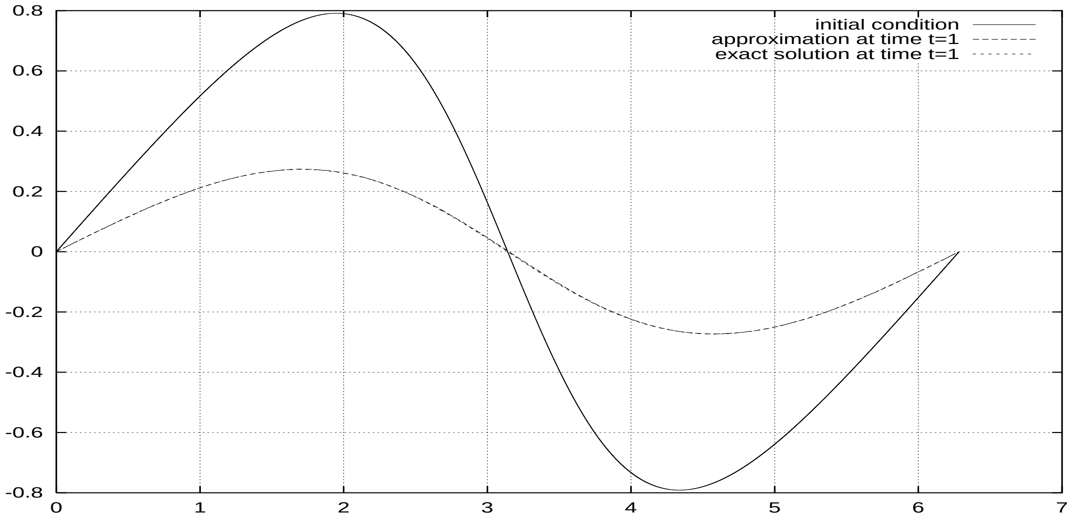

Figure 1: *Exact and numerical solutions of (6) obtained at time* $t = 1$, *for* 10^6 *particles and* $\Delta t = 10^{-2}$ *with the Lépingle scheme.*

results in case the spatial domain is $\mathbb{R}$), we concentrate our numerical study on the dependence on the time step. That is why we take a large number of particles $N = 10^6$. According to Theorem 2.2, $\mathbb{E}\|v(1,.) - \overline{V}(1,.)\|_{L^1([0,2\pi])} \leq 2\pi \sup_{x\in[0,2\pi]} \mathbb{E}|v(1,x) - \overline{V}(1,x)| \leq C(\Delta t + N^{-1/2})$. Since it is not possible to compute the last quantity, we compute the first one by averaging $\|v(1,.) - \overline{V}(1,.)\|_{L^1([0,2\pi])}$ over 20 runs of the particle method and give the dependence of the result on Δt in Table 1 and Figure 2.

We need to check that our test case (6) produces a significant rate of effective reflections. If this rate is too small, we only observe the effect of the classical Euler scheme (without reflection) with weak convergence also in Δt, and we cannot conclude on the convergence of the Lépingle scheme. The rate of effective reflections is around 10% for this test case : more precisely there are about 10% of the particles in $[0, \alpha_0] \cup [\alpha_1, 2\pi]$ at each time-step. For these particles, we compute the correction term C in (5). When we discretize the particle system according to the projected Euler scheme, which treats the reflection simply by projection onto $[0, 1]$, we clearly observe a sublinear convergence in Δt (see Table 1 and Figure 2). The projected Euler scheme does not use the correction term C whatever the position of the particle and its weak convergence rate is in $\mathcal{O}(\Delta t^{1/2})$, (see [6]). Therefore we can conclude that the quasi-linear decreasing of the error for the Lépingle scheme confirms our theoretical analysis.

Δt	Lépingle scheme	Confidence interval at 95%	Projection scheme	Confidence interval at 95%
2^{-1}	0.0940	[0.0933,0.0946]	0.2510	[0.2501,0.2519]
2^{-2}	0.0585	[0.0579,0.0591]	0.2320	[0.2309,0.2329]
2^{-3}	0.0329	[0.0322,0.0336]	0.1964	[0.1953,0.1975]
2^{-4}	0.0173	[0.0166,0.0180]	0.1568	[0.1557,0.1578]
2^{-5}	0.0083	[0.0076,0.0090]	0.1241	[0.1227,0.1254]
2^{-6}	0.0053	[0.0045,0.0060]	0.0982	[0.0969,0.0995]
2^{-7}	0.0049	[0.0043,0.0055]	0.0779	[0.0765,0.0793]
2^{-8}	0.0050	[0.0042,0.0058]	0.0635	[0.0627,0.0643]

Table 1: *Expectation of* L^1 *norm of error at* $t = 1$ *for* $N = 10^6$ *particles (*$\|v(\cdot, 1)\|_{L^1([0,2\pi])} = 1.09$*)*

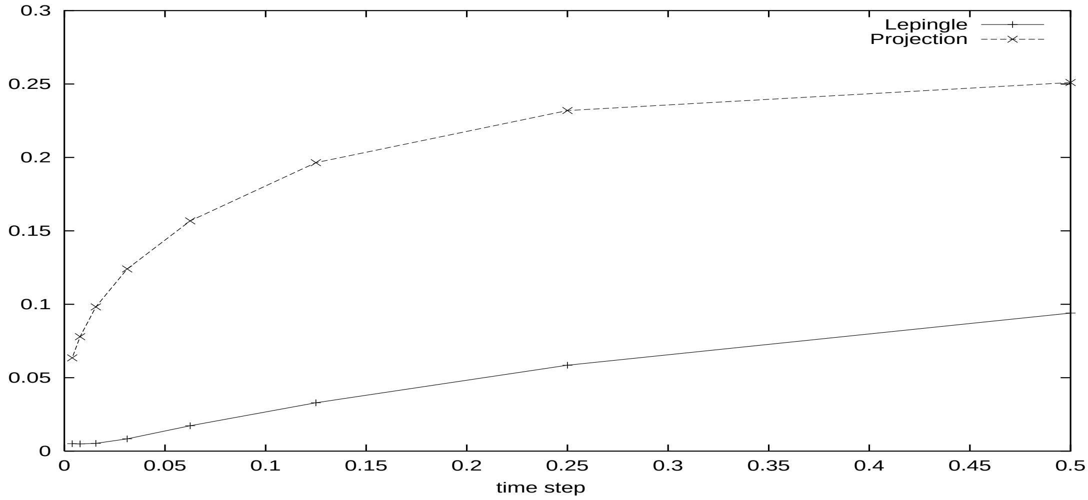

Figure 2: $\mathbb{E}\|v(\cdot,1)-\overline{V}(\cdot,1)\|_{L^1(\mathbb{R})}$ *in terms of* Δt *(*$N=10^6$*).*

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